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Quantum charged spinning particles in a strong magnetic field (a quantal guiding centre theory)

P Maraner

Center for Theoretical Physics and INFN, Massachusetts Institute of Technology, Cambridge MA 02139-437, USA

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Abstract. A *quantal guiding centre theory* is presented which allows a systematical study of the separation of the different time scale behaviours of a quantum charged spinning particle moving in an external inhomogeneous magnetic field. A suitable set of operators adapting to the canonical structure of the problem and generalizing the kinematical momenta and guiding centre operators of a particle coupled to a homogeneous magnetic field is constructed. This allows us to rewrite the Pauli Hamiltonian as a power series in the magnetic length $l_B = \sqrt{\hbar c/eB}$ making the problem amenable to a perturbative analysis. The first two terms of the series are explicitly constructed. The effective adiabatic dynamics turns out to be in coupling with a gauge field and a scalar potential. The mechanism producing such magnetic-induced geometric-magnetism is investigated in some detail.

1. Introduction

The motion of a charged particle in a strong inhomogeneous magnetic field is a nontrivial problem displaying a variety of very interesting phenomena ranging from chaos to phase anholonomy. Being of utmost importance in plasma physics, especially in the study of magnetic confinement, the subject has been worked out in great detail in classical mechanics with special attention to phenomenological implications as well as to formal aspects. The canonical structure of the problem, in particular, has been deeply investigated only relatively recently time by Littlejohn [1], revealing the appearance of geometry induced gauge structures in the adiabatic motion of classical charged particles. Very little, on the other hand, is known about the behaviour of quantum particles in strong inhomogeneous magnetic fields. The reason is essentially that the techniques developed for classical mechanics do not easily generalize to the quantum context. Some work has been done for neutral spinning particles by Berry [2], Aharonov and Stern [3] and Littlejohn and Weigert [4] in connection with geometrical phases, whereas a quantum treatment for charged spinning particles is still missing. It is the purpose of this paper to present a *quantal guiding centre theory* where the coupling between the spin and spatial degrees of freedom of a quantum charged spinning particle moving in a strong inhomogeneous magnetic field is systematically taken into account. This allows us to extend to the quantum domain the previous classical results. Our treatment, algebraic in nature, is a re-elaboration and a simplification of the technique originally proposed by Littlejohn in classical mechanics. It is based on a different choice of non-canonical variables adapting to classical as well as quantum mechanics. Depending essentially on the canonical structure the method applies indistinctly to both theories. Nevertheless, focus on the quantum problem.

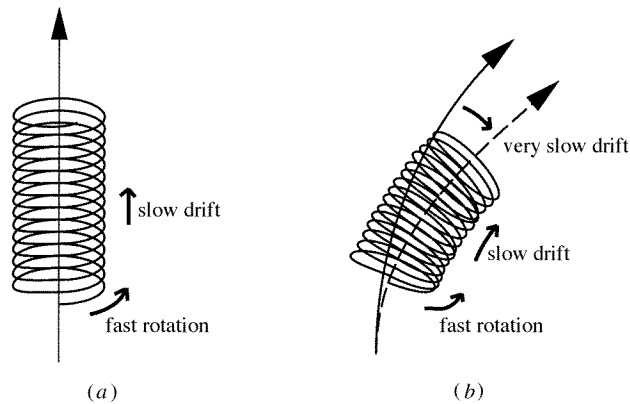


Figure 1. Different time scale behaviours of a charged particle in a strong magnetic field: (a) *fast rotation* of the particle and *slow* guiding centre drift in a homogeneous field; (b) in the inhomogeneous case the guiding centre drifts away from the field line *very slowly*.

In order to better understand the strong-field regime of a quantum particle moving in an external magnetic field it is helpful to have in mind the main features of the equivalent classical problem [5]. Let us therefore consider a classical particle of mass m and charge e moving in a homogeneous magnetic field of strength B . As is well known the trajectory of the particle is represented by a helix wrapping around a field line, as sketched in figure 1(a) the particle performs a uniform circular motion of frequency $\omega_B = eB/mc$ and radius $r_B = mc|v_\perp|/eB$ ($|v_\perp|$ is the norm of the normal component of the velocity) in the directions normal to the field, while the centre of the orbit, called the *guiding centre*, moves freely along a field line. Keeping fixed the initial condition, the stronger the magnetic field the faster the rotation of the particle when compared with the drift along the field direction and the smaller the portion of space explored by the particle around the field line. This indicates the presence of different time scales in the dynamics of the system and gives the reason why the motion in a very strong magnetic field may be studied along the same lines as that in a weakly inhomogeneous one. Let us introduce now a small inhomogeneity in the field. The picture of the motion should not change substantially. The particle keeps on rotating around its guiding centre while the frequency and the radius now weakly depend on the position. The guiding centre still drifts along a field line. In this case, however, the guiding centre does not remain exactly on a single field line. It starts drifting very slowly in the direction normal to the field. Three different time scale behaviours of the system may therefore be distinguished: the *fast rotation* of the particle around the guiding centre, the *slow drift* of the guiding centre along a magnetic field line and the *very slow drift* of the guiding centre in the direction normal to the field. The situation is sketched in figure 1(b). For stronger magnetic fields the separation of these three degrees of freedom becomes more clear.

An outlook to the canonical structure of the homogeneous case immediately makes clear how the introduction of kinematical momenta and guiding centre operators allows the description of these three degrees of freedom. This is briefly reported in section 2 where the relevant notations of the paper are also set up. After discussing the geometrical complications involved in the adiabatic motion of a charged particle in an inhomogeneous magnetic field, section 3, an appropriate set of non-canonical operators generalizing the one used in the discussion of the homogeneous problem is constructed in section 4. These

are obtained as formal power series in the magnetic length $l_B = \sqrt{\hbar c/eB}$, which appears naturally as the adiabatic parameter of the theory. The Pauli Hamiltonian describing the motion of the particle is then rewritten in terms of the new adiabatic operators in sections 5 and 6, whereas the anholonomic effects appearing in the adiabatic separation of the fast and slow/very slow degrees of freedom are discussed in section 7. Our results are summarized in equations (42), (43) and (44). In the classical limit these reproduce correctly the classical theory. Section 8 contains our conclusions.

2. Canonical structure of the guiding centre motion

Magnetic interactions appear essentially as modifications of the canonical structure of a dynamical system. It is worthwhile to start by briefly discussing this peculiarity in the elementary case of a quantum charged spinning particle in a homogeneous magnetic field. This allows us to immediately focus on the heart of the problem establishing at the same time terminology and notations. We consider, therefore, a spin- $\frac{1}{2}$ particle of mass m , charge e and gyromagnetic factor g moving in space under the influence of the *homogeneous* field $\mathbf{B}(\mathbf{x}) = B\hat{z}$. As in the inhomogeneous case, to be discussed later on, the physical dimension of the magnetic field is reabsorbed in the scale factor B . $B^{-1/2}$, appropriately rescaled, will play the role of the adiabatic parameter of our theory. Introducing an arbitrary choice of the vector potential \mathbf{a} for the dimensionless field $\mathbf{B}(\mathbf{x})/B$, $\text{rot } \mathbf{a} = \hat{z}$, the motion of the particle is described by the Pauli Hamiltonian

$$\mathcal{H} = \frac{1}{2m} \left(-i\hbar\nabla - \frac{eB}{c}\mathbf{a} \right)^2 + g \frac{\hbar eB}{mc} \hat{z} \cdot \boldsymbol{\sigma} \quad (1)$$

$\nabla = (\partial_x, \partial_y, \partial_z)$ denoting the gradient operator and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ the matrix-valued vector constructed by means of the three Pauli matrices. As is well known the solution of this simple problem was first obtained by Landau at the beginning of the thirties and leads naturally to replace the standard set of canonical operators $p_i = -i\hbar\partial_i$, x^i , $i = 1, 2, 3$, by the gauge invariant *kinematical momenta* $\pi_i = p_i - (eB/c)a_i$ and the *guiding centre operators* $X^i = x^i + (c/eB)\epsilon^{ij}\pi_j$. A straightforward computation yields the nonvanishing commutation relation among the new variables

$$[\pi_2, \pi_1] = -i\frac{\hbar eB}{c} \quad [\pi_3, X^3] = -i\hbar \quad [X^1, X^2] = -i\frac{\hbar c}{eB} \quad (2)$$

indicating π_2 - π_1 , π_3 - X^3 and X^1 - X^2 as couples of conjugates variables. Moreover, the scale dependence of the commutators (2) allows us to identify the three pairs of operators as describing respectively the *fast*, the *slow* and the *very slow* degrees of freedom of the system (see e.g. [6]). In terms of the new gauge invariant operators Hamiltonian (1) can be rewritten in the very simple form $\mathcal{H} = (\pi_1^2 + \pi_2^2 + \pi_3^2)/2m + g\hbar eB\sigma_3/mc$. The harmonic oscillator term $(\pi_1^2 + \pi_2^2)/2m$ takes into account the rapid rotation of the particle around its guiding centre. The free term $\pi_3^2/2m$ produces the slow drift of the guiding centre along the straight magnetic field lines. As the very slow variables X^1 and X^2 are constants of motion, the guiding centre does not move in the directions normal to the field. Let us stress that in the canonical formalism the spatial rotation of the particle around its guiding centre is taken into account by the phase space trajectory of a couple of conjugate variables: the particle's velocity components π_1 and π_2 in the directions normal to the field. The presence of an external magnetic field therefore produces a rotation of the canonical structure, mixing up spatial coordinates and canonical momenta in new canonical operators adapting to the different time scale behaviours of the particle! In section 4 we will construct

such ‘adapted operators’—as power series in the adiabatic parameter—for the motion of a quantum charged spinning particle in an arbitrary magnetic field. This allows the extension of the Hamiltonian approach developed by Littlejohn [1] to quantum mechanics. The case of a magnetic field with constant direction has been previously considered in [6]. Some preparatory material is, however, necessary.

First of all it is convenient to introduce dimensionless quantities by factorizing the energy scale $\hbar\omega_B$, $\omega_B = eB/mc$, from the Hamiltonian. This leads one to redefine kinematical momenta and guiding centre operators as

$$\pi_i = -il_B\partial_i - l_B^{-1}a_i(\mathbf{x}) \quad (3)$$

$$X^i = x^i + l_B\epsilon^{ij}\pi_j \quad (4)$$

$l_B = \sqrt{\hbar c/eB}$ being the *magnetic length*. The relevant commutation relations may then be recast in the compact and very convenient form

$$\left. \begin{aligned} [\pi_i, \pi_j] &= i\epsilon_{ij} \\ [\sigma_i, \sigma_j] &= i\epsilon_{ijk}\sigma_k \end{aligned} \right\} \quad \textit{fast} \quad (5)$$

$$[\pi_i, X^j] = -il_B\delta_i^3\delta_3^j \quad \textit{slow}$$

$$[X^i, X^j] = -il_B^2\epsilon^{ij} \quad \textit{very slow}$$

where the spin variables have also been considered.

A second and more serious task is the discussion of the geometrical structure responsible for the anholonomic effects appearing in the adiabatic motion in a strong magnetic field.

3. Magnetism and geometric-magnetism

The beautiful analysis of the adiabatic separation of *fast* and *slow* degrees of freedom in a quantum system proposed by Berry [7], Kuratsuji and Iida [8], Moody *et al* [9], Jackiw [10] and others, has pointed out that in lowest order the reaction of the fast to the slow dynamics is through a geometry-induced gauge structure resembling that of (electro-) magnetism. This phenomenon has been identified and found to be important in a variety of physical contexts [11] and has been recently referred to by Berry and Robbins [12] as *geometric-magnetism*. A rather curious fact, first pointed out by Littlejohn [1] in a series of papers on the canonical structure of classical guiding centre motion, is that, in some circumstances, magnetism itself may generate geometric-magnetism. The aim of the present section is to discuss the geometry involved in such ‘magnetic-induced geometric-magnetism’.

It is useful to begin by briefly recalling the geometrical character of the kinematical quantities characterizing the motion of a particle in space. This will lead to a rather intuitive picture of the geometrical structure involved in the adiabatic motion of a charged spinning particle in a strong magnetic field, allowing us at the same time to frame it in a general and rigorous context. The state of a particle moving in space is completely characterized by its position \mathbf{x} and its velocity \mathbf{v} , i.e. by a point in the *tangent bundle* TR^3 of the three-dimensional Euclidean space R^3 . The flat parallel transport of R^3 makes it natural to parameterize every fibre T_xR^3 of the bundle by means of a fixed reference frame in R^3 , that is, to identify the tangent space in every point \mathbf{x} with the physical space itself. Such an identification is certainly very useful in most circumstances, but it is just a convention. In principle we are free to choose arbitrarily the frame of T_xR^3 in every \mathbf{x} . The parallel transport—and not the way we describe it—is all that matters. This freedom of arbitrarily rotating the reference frame of the tangent space in every point \mathbf{x} consists a local $SO(3)$ symmetry and plays a crucial role in what follows. To visualize the situation, therefore, we

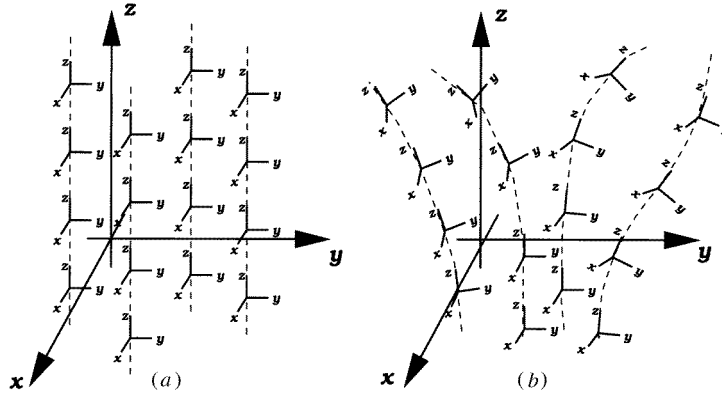


Figure 2. Framing the tangent bundle TR^3 of the physical space: (a) by means of a single fixed frame in R^3 ; (b) by using local reference frames adapting to the magnetic field lines geometry.

shall picture the Euclidean space as filled up with orthonormal reference frames. To start with, we can imagine all of them as being combed parallel to a single fixed frame $\{\hat{x}, \hat{y}, \hat{z}\}$ in R^3 (see figure 2(a)), but this is not always the best choice even for the flat case.

3.1. The magnetic line bundle

As qualitatively sketched above, the motion of a charged spinning particle in a strong magnetic field is characterized by the separation of time scales in the three degrees of freedom, making the system amenable to a perturbative analysis. In the lowest order of approximation the particle performs a *fast* rotation in the plane normal to the field line at which its guiding centre is located. This is taken into account by the two components normal to the field of the particle's velocity (to this order a couple of conjugate variables). Disregarding the *slow* drift of the guiding centre along the field line and the *very slow* motion, the velocity of a particle whose guiding centre is located in \mathbf{x} , is effectively constrained to the plane $\mu_{\mathbf{x}}$ generated by the vectors normal to the field in \mathbf{x} . In every point of the space the magnetic field $\mathbf{b}(\mathbf{x})$ picks the complex line $\mu_{\mathbf{x}}$ out of the tangent space $T_{\mathbf{x}}R^3$, reducing the *tangent bundle* TR^3 to a complex line bundle, hereafter the *magnetic line bundle* \mathcal{M}^\dagger . It is then natural to use the local $SO(3)$ symmetry of the theory to adapt the parameterization of TR^3 to the sub-bundle \mathcal{M} by combing, say, the \hat{z} direction of the frame of every $T_{\mathbf{x}}R^3$ according to the direction of the field. We so smoothly introduce point-dependent adapted reference frames $\{e_1, e_2, e_3\}$ in such a way that in every point \mathbf{x} the vectors $e_1(\mathbf{x}), e_2(\mathbf{x})$ generate $\mu_{\mathbf{x}}$ while $e_3(\mathbf{x})$ is aligned with $\mathbf{b}(\mathbf{x})$ (see figure 2(b)). Such reference frames are commonly used in the discussion of geometrically non-trivial physical problems such as in general relativity and are referred to as *anholonomic frames*. It is worthwhile to note that fixing e_3 according to the field direction reduces the local $SO(3)$ symmetry of TR^3 into the local $SO(2) \equiv U(1)$ symmetry of \mathcal{M} . The vectors $e_1(\mathbf{x})$ and $e_2(\mathbf{x})$ are in fact determined up to the rotation

$$\begin{aligned} e_1(\mathbf{x}) &\rightarrow e_1(\mathbf{x}) \cos \chi(\mathbf{x}) - e_2(\mathbf{x}) \sin \chi(\mathbf{x}) \\ e_2(\mathbf{x}) &\rightarrow e_1(\mathbf{x}) \sin \chi(\mathbf{x}) + e_2(\mathbf{x}) \cos \chi(\mathbf{x}) \end{aligned} \tag{6}$$

[†] This sub-bundle of TR^3 may be identified with the *plane bundle* of Felsager and Leinaas [13]. See also the related paper of Gliozzi [14].

where $\chi(\mathbf{x})$ is a point-dependent angle. This residual ambiguity will result in the gauge freedom of our theory.

3.2. Magnetic line bundle geometry

We may now wonder how the vectors lying in \mathcal{M} are transported from point to point, that is, whether the geometry of the magnetic line bundle is trivial or not. To this task we proceed in two steps. Considering a vector $\mathbf{w}(\mathbf{x}) = w^\nu e_\nu(\mathbf{x})$, $\nu = 1, 2$, in $\mu_{\mathbf{x}}$, we first transport it from the point \mathbf{x} to the infinitesimally closest point $\mathbf{x} + d\mathbf{x}$ by means of the Euclidean parallel transport of R^3 and, second, we project it onto the plane $\mu_{\mathbf{x}+d\mathbf{x}}$. (i) The Euclidean parallel transport of \mathbf{w} in $\mathbf{x} + d\mathbf{x}$ may be immediately evaluated as

$$\mathbf{w}(\mathbf{x} + d\mathbf{x}) = \mathbf{w}(\mathbf{x}) - w^\nu (e_\nu \cdot \partial_k e_i) dx^k e_i$$

the Roman indices running over 1, 2, 3, and the Greek indices over 1, 2 and where the sum over repeated indices is implied[†]. The three quantities[‡] $e_1 \cdot \partial_k e_2$, $e_1 \cdot \partial_k e_3$ and $e_2 \cdot \partial_k e_3$ characterize the flat parallel transport of R^3 in the anholonomic frame. (It is in fact possible to make them vanish by rotating the adapted frames $\{e_1, e_2, e_3\}$ back to fixed directions in every point.) (ii) The projection onto $\mu_{\mathbf{x}+d\mathbf{x}}$ then yields

$$\mathbf{w}(\mathbf{x} + d\mathbf{x})|_\mu = \mathbf{w}(\mathbf{x}) - w^\mu (e_1 \cdot \partial_k e_2) dx^k \varepsilon_\mu^\nu e_\nu$$

indicating that the parallel transport along the infinitesimal path connecting \mathbf{x} to $\mathbf{x} + d\mathbf{x}$ produces the vector \mathbf{w} to be rotated by the infinitesimal angle $d\alpha = (e_1 \cdot \partial_k e_2) dx^k$. When parallel transported along a finite closed path γ the vector will therefore return to the starting point rotated by the angle [13]

$$\alpha_\gamma = \oint_\gamma (e_1 \cdot \partial_k e_2) dx^k.$$

As this quantity is in general different from zero, the geometry of the magnetic line bundle is not flat. The operation of locally projecting onto the plane μ reduces the trivial $SO(3)$ local symmetry of the theory to a non-trivial $SO(2) \equiv U(1)$ local symmetry! This local structure is described by a magnetic-like $U(1)$ gauge theory. The parallel transport of the magnetic line bundle \mathcal{M} , is in fact completely characterized by the vector

$$\mathcal{A}_k = e_1 \cdot \partial_k e_2 \tag{7}$$

which is the *connection 1-form* of \mathcal{M} . \mathcal{A} appears in the theory as a geometry-induced vector potential (not to be confused with the vector potential \mathbf{a} representing the real magnetic field \mathbf{b}). A point-dependent redefinition of the local basis $\{e_1(\mathbf{x}), e_2(\mathbf{x})\}$ plays the same role of a gauge transformation. Under the rotation (6) the vector (7) transforms according to $\mathcal{A}_k \rightarrow \mathcal{A}_k + \partial_k \chi$. The associate geometry-induced magnetic field $\mathcal{B}_k = \varepsilon_{kmn} \mathcal{B}_{mn}$, $\mathcal{B}_{mn} = \partial_m \mathcal{A}_n - \partial_n \mathcal{A}_m$, the *curvature 2-form* of \mathcal{M} , may also be considered. It is obviously a gauge-invariant quantity and, being the rotor of a vector field, satisfies the Bianchi identity $\text{div } \mathcal{B} = 0$.

While the geometry-induced vector potential \mathcal{A} completely characterizes the intrinsic geometry of the magnetic line bundle \mathcal{M} , the other two quantities

$$\begin{aligned} l_{1k} &= e_1 \cdot \partial_k e_3 \\ l_{2k} &= e_2 \cdot \partial_k e_3 \end{aligned} \tag{8}$$

[†] This notation will be employed throughout the rest of this paper.

[‡] The vectors e_1 , e_2 and e_3 being orthonormal in every point \mathbf{x} , $e_i \cdot e_j = \delta_{ij}$, these are the only independent quantities.

describing the flat parallel transport of R^3 in the anholonomic frame $\{e_1, e_2, e_3\}$, takes into account its extrinsic geometry. Since the curvature of the tangent bundle TR^3 is zero the three quantities \mathcal{A} , l_1 and l_2 are obviously not independent. They are related by the equivalent of the Gauss, Codazzi-Mainardi and Ricci equations. The latter, as an example, allows us to re-express the geometry-induced gauge field \mathcal{B} entirely in terms of l_1 and l_2 as

$$\mathcal{B} = l_1 \wedge l_2 \tag{9}$$

where $\mathcal{B}_{kl} = (l_{1k}l_{2l} - l_{1l}l_{2k})/2$, and \wedge indicates the external product of R^3 [13]. With respect to the point-dependent rotation (6) l_1 and l_2 transform as vectors ($l_1 \rightarrow l_1 \cos \chi - l_2 \sin \chi$, $l_2 \rightarrow l_1 \sin \chi + l_2 \cos \chi$) making the gauge invariance of \mathcal{B} manifest.

3.3. Magnetic field lines geometry

Though the geometry of a magnetic field is completely characterized by two independent functions (e.g. the two independent components of the real magnetic field \mathbf{b} , or of the geometry-induced magnetic field \mathcal{B} , etc) it may be useful to look at the problem from different points of view. We may wonder, as an example, how the intrinsic/extrinsic geometry of the line bundle \mathcal{M} is related to the geometry of magnetic field lines. To this task we start by observing that the projection along the field direction of the two vectors l_1, l_2 may be identified with the two *second fundamental forms* of the embedding of the magnetic field lines in R^3 [15]. In every point of the space the curvature k of the magnetic field line going through that point may therefore be expressed as

$$k = \sqrt{(e_3 \cdot l_1)^2 + (e_3 \cdot l_2)^2}. \tag{10}$$

In a similar way the projections along the field direction of the geometry-induced vector potential \mathcal{A} have to be identified with the *normal fundamental form* of the embedding of the field lines in R^3 (i.e. with the connection form induced by the Euclidean geometry onto the normal bundle of every field line) [15]. Up to the gradient of an arbitrary function—representing again the freedom of arbitrarily rotating the reference frame in the normal planes—in every point of the space the torsion τ of the magnetic field line going through that point may be written as

$$\tau = e_3 \cdot \mathcal{A}. \tag{11}$$

Curvature and torsion completely characterize the geometry of every single magnetic field line and contain, in principle, all the information relative to the geometry of our problem. On the other hand we may also wonder about the global properties of the foliation of R^3 in terms of field lines. Of particular relevance for the adiabatic motion of a particle in an external magnetic field is the possibility of foliating space by means of surfaces everywhere orthogonal to the field lines. By virtue of the Frobenius theorem this is controlled by the vanishing of the scalar $\mathcal{F} = e_3 \cdot \text{rot } e_3$. In terms of the magnetic line bundle geometry

$$\mathcal{F} = e_1 \cdot l_2 - e_2 \cdot l_1. \tag{12}$$

The magnetic field lines torsion τ and the Frobenius invariant \mathcal{F} play a crucial role in the description of the anholonomic effects appearing in the adiabatic motion of a charged particle in a strong magnetic field.

4. Adiabatic quantum variables

We are now ready for the construction of a set of adiabatic operators adapting to the different time scale behaviours of a quantum particle in a strong, but otherwise arbitrary, magnetic field. Let us consider therefore a spin- $\frac{1}{2}$ particle of mass m , charge e and gyromagnetic factor g moving in space under the influence of the *inhomogeneous* magnetic field $\mathbf{B}(\mathbf{x}) = B\mathbf{b}(\mathbf{x})$, the physical dimension of the field being again re-absorbed in the scale factor B . Denoting by \mathbf{a} an arbitrary choice of the vector potential, $\text{rot } \mathbf{a} = \mathbf{b}$, the dynamics of the system is described by the Pauli Hamiltonian

$$\mathcal{H}/\hbar\omega_B = \frac{1}{2}\pi_i\pi_i + gb_i(\mathbf{x})\sigma_i \quad (13)$$

where the kinematical momenta $\pi_i = -i l_B \partial_i - a_i(\mathbf{x})/l_B$ have been introduced. The inhomogeneity of the magnetic field makes Hamiltonian (13) dependent on the position operators \mathbf{x} , explicitly through spin term $gb_i(\mathbf{x})\sigma_i$ and implicitly through the commutation relations of the π_i s. In spite of the simple quadratic dependence of (13) on the kinematical momenta, π_1 and π_2 are in fact no longer conjugate variables and neither commute with π_3 : the set of operators $\{\pi_i, x^i; i = 1, 2, 3\}$ fulfil the commutation relations

$$[\pi_i, \pi_j] = i b_{ij}(\mathbf{x}) \quad [\pi_i, x^j] = -i l_B \delta_i^j \quad [x^i, x^j] = 0 \quad (14)$$

$b_{ij}(\mathbf{x}) = \varepsilon_{ijk} b_k(\mathbf{x})$ denoting the skew-symmetric 2-form associated to the field. In the lowest approximation we nevertheless expect the relevant degree of freedom of the system to be taken into account by the two components of the particle's velocity normal to the field. Considering the position operators x^i s as adiabatic parameters driving the fast motion of the system we expect, therefore, the rapid rotation of the particle around its guiding centre to be separated from the slow and very slow motion by simply referring the kinematical momenta to the adapted frames introduced in the previous section. For the sake of concreteness we shall indicate by $R_i^j(\mathbf{x})$ the point-dependent rotation bringing the fixed frame $\{\hat{x}, \hat{y}, \hat{z}\}$ into the adapted frame $\{e_1(\mathbf{x}), e_2(\mathbf{x}), e_3(\mathbf{x})\}$. This allows us to decompose the field $\mathbf{b}(\mathbf{x})$ in terms of its norm $b = \sqrt{\mathbf{b} \cdot \mathbf{b}}$ and its direction $\mathbf{b}/b = R_i^j \hat{z}_j$ as $b_i(\mathbf{x}) = b(\mathbf{x}) R_i^j(\mathbf{x}) \hat{z}_j$. Once the rotation has been performed the kinematical momentum along the field direction decouples, up to higher order terms in the adiabatic parameter l_B , from the other two components. The commutator of these, on the other hand, is proportional to $b(\mathbf{x})$. Stated in a different way, in the adapted frame the particle sees an effective magnetic field of constant direction and intensity $b(\mathbf{x})$. To make the velocity components normal to the field in a couple of conjugate operators it is now sufficient to rescale them by the point-dependent factor $b^{-1/2}(\mathbf{x})$ (see [6]). We shall indicate by $D_i^j(\mathbf{x})$ the point-dependent dilatation $D_i^j = \text{diag}(b^{1/2}, b^{1/2}, 1)$ rescaling the first and second components of a vector by $b^{1/2}$ and leaving the third one unchanged.

In order to construct operators adapting to the *fast* time scale behaviour of the system two point-dependent operations have to be performed: (i) a rotation $R_i^j(\mathbf{x})$ to the local adapted frame and (ii) a dilatation $D_i^j(\mathbf{x})$ rescaling the normal components of the kinematical momenta. As the particle coordinates are not external parameters but dynamical variables of the problem, these operations will produce higher order corrections in the various commutators. We shall therefore proceed order by order in the adiabatic parameter l_B by constructing sets of adiabatic operators fulfilling the desired commutation relation up to a given order in l_B : at the n th order we shall look for a set of operators $\{\Pi_i^{(n)}, X_{(n)}^i; i = 1, 2, 3\}$ fulfilling the conditions:

- $\Pi_1^{(n)}, \Pi_2^{(n)}$ are a couple of conjugate operators up to terms of order l_B^n
- $\Pi_3^{(n)}, X_{(n)}^1, X_{(n)}^2, X_{(n)}^3$ commute with $\Pi_1^{(n)}, \Pi_2^{(n)}$ up to terms of order l_B^n

• in the limit of a homogeneous field, $\mathbf{b}(\mathbf{x}) \rightarrow \hat{\mathbf{z}}$, the adiabatic kinematical momenta $\Pi_i^{(n)}$ s and guiding centre operators $X_{(n)}^i$ s should reduce to the expressions (3) and (4) respectively.

As our present task is to separate the fast degree of freedom from the slow and very slow motion, we do not insist for the moment $X_{(n)}^3 - \Pi_3^{(n)}$ and $X_{(n)}^1 - X_{(n)}^2$ to be pairs of conjugate operators as in the homogeneous case.

For computational purposes it is very convenient to use a compact notation which does not distinguish among the physical unequivalent directions along and normal to the field. This probably obscures for a while the physical contents of the various expression but greatly simplifies formal manipulations. When necessary we will expand the notation in order to shed light on the physics involved. For the moment we proceed in the opposite direction by introducing the point-dependent matrix

$$\beta_i^j(\mathbf{x}) = D_i^{-1k}(\mathbf{x})R_k^{-1j}(\mathbf{x}) \tag{15}$$

representing the successive application of the two operations necessary to construct the adapted kinematical momenta in the lowest order. This allows us to rewrite the skew-symmetric 2-form $b_{ij}(\mathbf{x})$ in terms of $\varepsilon_{kl} = \varepsilon_{kl3}$ (representing a homogeneous field directed along \mathbf{z}):

$$b_{ij}(\mathbf{x}) = \beta_i^{-1k}(\mathbf{x})\beta_j^{-1l}(\mathbf{x})\varepsilon_{kl}. \tag{16}$$

The matrix β_i^j and this representation of the field are very useful in the construction of the adiabatic quantum variables.

4.1. Zero-order operators

In order to construct the zero-order operators fulfilling the desired conditions up to terms of order l_B it is sufficient to operate the rotation and the dilatation discussed above:

$$\Pi_i^{(0)} = \frac{1}{2}\{\beta_i^k, \pi_k\} \tag{17}$$

the matrix β_i^k being evaluated in $\mathbf{X}_{(0)} \equiv \mathbf{x}$. The anticommutator $\{, \}$ is obviously introduced in order to make the $\Pi_i^{(0)}$ s Hermitian. A rapid computation confirms our deductions yielding the commutation relations fulfilled by the zero-order adiabatic operators as

$$\begin{aligned} [\Pi_i^{(0)}, \Pi_j^{(0)}] &= i\varepsilon_{ij} - i\frac{l_B}{2}\varepsilon_{ijh}\varepsilon^{hkl}\{\beta_k^m\Gamma_{ml}^n, \Pi_n^{(0)}\} \\ [\Pi_i^{(0)}, X_{(0)}^j] &= -il_B\beta_i^j \\ [X_{(0)}^i, X_{(0)}^j] &= 0 \end{aligned} \tag{18}$$

where $\Gamma_{ki}^j = (\partial_k\beta_i^h)\beta_h^{-1j}$ and all the functions are evaluated in $\mathbf{X}_{(0)}$. $\Pi_1^{(0)}$ and $\Pi_2^{(0)}$ are conjugate operators up to $\mathcal{O}(l_B)$. The commutators depend on the derivative of magnetic field through the vector-valued matrix

$$(\Gamma_k)_i^j = \begin{pmatrix} -\frac{1}{2}\frac{\partial_k b}{b} & -\mathcal{A}_k & -b^{-1/2}l_{1k} \\ \mathcal{A}_k & -\frac{1}{2}\frac{\partial_k b}{b} & -b^{-1/2}l_{2k} \\ b^{1/2}l_{1k} & b^{1/2}l_{2k} & 0 \end{pmatrix} \tag{19}$$

allowing us to clearly distinguish the effects produced by a variation of the norm of the magnetic field from that produced by a change of direction. The latter are entirely geometrical in character as they are taken into account by the magnetic line bundle connection form \mathcal{A} and by the two extrinsic vectors l_1 and l_2 .

4.2. *First-order operators*

Whereas the construction of the zero-order operators is in some way suggested by the physics of the problem, a more technical effort is required for higher-order terms. The form of the first-order guiding centre operators is nevertheless suggested by the corresponding homogeneous expression (4),

$$X_{(1)}^i = X_{(0)}^i + \frac{l_B}{2} \varepsilon^{kl} \{\beta_k^i, \Pi_l^{(0)}\} \tag{20}$$

the matrix β_k^i being again evaluated in $\mathbf{X}_{(0)}$. We immediately obtain the new commutation relation

$$\begin{aligned} [\Pi_i^{(0)}, \Pi_j^{(0)}] &= i\varepsilon_{ij} - i\frac{l_B}{2} \varepsilon_{ijh} \varepsilon^{hkl} \{\beta_k^m \Gamma_{ml}^n, \Pi_n^{(0)}\} + \mathcal{O}(l_B^2) \\ [\Pi_i^{(0)}, X_{(1)}^j] &= -il_B \delta_i^3 \beta_3^j + \mathcal{O}(l_B^2) \\ [X_{(1)}^i, X_{(1)}^j] &= -il_B^2 \varepsilon^{kl} \beta_k^i \beta_l^j + \mathcal{O}(l_B^3) \end{aligned} \tag{21}$$

indicating the $\mathcal{O}(l_B^2)$ decoupling of the adiabatic guiding centre operators from $\Pi_1^{(0)}$ and $\Pi_2^{(0)}$. All the functions are now evaluated in $\mathbf{X}_{(1)}$. Though our analysis will be carried out up to $\mathcal{O}(l_B^2)$, we also wrote the the first nonvanishing contribution to the commutators among the $X_{(1)}^i$ s, which is of order l_B^2 . Even if it is unimportant for the present calculation, it allows us to visualize the very slow time scale of the system.

The construction of the first-order kinematical momenta is performed by looking for order l_B counterterms to be added to the $\Pi_i^{(0)}$ s. These should be homogeneous second-order polynomials in the $\Pi_i^{(0)}$ s with coefficients depending on $\mathbf{X}_{(1)}$. A rather tedious computation produces

$$\Pi_i^{(1)} = \Pi_i^{(0)} + l_B c_{ij}^{klmn} \{\beta_m^h \Gamma_{hn}^j, \{\Pi_k^{(0)}, \Pi_l^{(0)}\}\} \tag{22}$$

where $c_{ij}^{klmn} = \frac{1}{24} \varepsilon_{ih} \varepsilon^{kh} (2\delta_j^l + \delta_j^3 \delta_3^l) \varepsilon^{mn} + \frac{1}{8} \delta_i^3 \varepsilon^{kh} (\delta_j^l + \delta_j^3 \delta_3^l) \varepsilon_{hg} \varepsilon^{gmn}$ and all the functions are evaluated in $\mathbf{X}_{(1)}$. When expanded these expressions do not look as complicated as at a first sight. We nevertheless insist on keeping this notation which greatly simplifies the following manipulations. The commutation relations among the first-order adiabatic variables are obtained as

$$\begin{aligned} [\Pi_i^{(1)}, \Pi_j^{(1)}] &= i\varepsilon_{ij} - i\frac{l_B}{4} \varepsilon_{ijk} \varepsilon^{kl} \left\{ \frac{\text{div } \mathbf{b}}{b}, \Pi_l^{(1)} \right\} + \mathcal{O}(l_B^2) \\ [\Pi_i^{(1)}, X_{(1)}^j] &= -il_B \delta_i^3 \beta_3^j + \mathcal{O}(l_B^2) \\ [X_{(1)}^i, X_{(1)}^j] &= -il_B^2 \varepsilon^{kl} \beta_k^i \beta_l^j + \mathcal{O}(l_B^3). \end{aligned} \tag{23}$$

It is very interesting to observe that a monopole singularity, that is a point of nonvanishing divergence, represents an obstruction in the construction of the adiabatic operators. Being concerned with real magnetic field we nevertheless assume $\text{div } \mathbf{b} = 0$ and carry on in our adiabatic analysis. $\Pi_1^{(1)}$ and $\Pi_2^{(1)}$ are then conjugate operators commuting with all the remaining variables up to terms of order l_B^2 and the fast degree of freedom decouples from the slow and very slow motion up to terms of this order.

4.3. *A non-canonical set of operators*

At least in principle it is possible to repeat this construction an arbitrary number of times getting, as power series in l_B , a set of adiabatic non-canonical operators $\{\Pi_i, X^i; i = 1, 2, 3\}$

fulfilling the commutation relations

$$[\Pi_i, \Pi_j] = i\varepsilon_{ij} \quad [\Pi_i, X^j] = -il_B \delta_i^3 R_3^{-1j} \quad [X^i, X^j] = -il_B^2 \varepsilon^{kl} b^{-1} R_k^{-1i} R_l^{-1j} \quad (24)$$

all the functions being now evaluated in \mathbf{X} . These formal series are in general—and have to be [16]—not convergent, representing nevertheless a very useful tool in the discussion of the adiabatic behaviour of the system. The description of the problem to a given order n in the adiabatic parameter l_B requires the knowledge of the first $n + 1$ terms of the adiabatic series, so that up to terms of order l_B^2 we may identify the Π_i s and X^i s with the $\Pi_i^{(1)}$ s and $X_{(1)}^i$ s respectively. An outlook to the commutation relation (24) allows us to clearly identify the dependence of the canonical structure on the variation of norm and direction of the magnetic field. Whereas a suitable redefinition of reference frames in TR^3 allows us to separate the fast degree of freedom from the others, the very slow variables are made into a pair of non-conjugate operators by an inhomogeneous intensity. A variation of the field direction even produces the mixing of very slow and slow variables. The description of these by means of pairs of conjugate operators requires the introduction of curvilinear coordinates in space [17], the so-called *Euler potentials* [18]. We do not insist further on this point, for the moment observing that under the action of Π_1, Π_2 and Π_3, X^1, X^2, X^3 the Hilbert space of the system separates into the direct sum of two subspaces describing respectively the rapid rotation of the particle and the guiding centre motion.

5. Expanding the Hamiltonian

The adiabatic operators Π and \mathbf{X} constructed in the previous section have been introduced in such a way as to embody the expected features of the motion of a quantum charged particle in a weakly inhomogeneous magnetic field. Their main advantage lies, in fact, in the very suitable form assumed by the Pauli Hamiltonian when rewritten in terms of these operators. To perform this task we first have to invert the power series expressing Π_i and X^i in terms of the operators π_i s and x^i s and, second, to replace these in (13). This yields the Hamiltonian as a power series in the magnetic length l_B ,

$$\mathcal{H} = \mathcal{H}^{(0)} + l_B \mathcal{H}^{(1)} + l_B^2 \mathcal{H}^{(2)} + \dots \quad (25)$$

allowing the adiabatic separation of the fast degree of freedom from the slow/very slow motion and the evaluation of approximate expressions of the spectrum and of the wavefunctions of the system. In order to get the π_i s and x^i s in terms of the Π_i s and X^i s we first recall that $X^i = X_{(1)}^i + \mathcal{O}(l_B^2)$. By rewriting $X_{(1)}^i$ in terms of the $\Pi_i^{(0)}$ s and $X_{(0)}^i = x^i$ s, equation (20), $\Pi_i^{(0)}$ in terms of the π_i s and x^i s (equation (17)), and by solving with respect to x^i , we then obtain x^i as a function of the π_i s and the X^i s, $x^i = x^i(\boldsymbol{\pi}, \mathbf{X})$. This allows us to rewrite $\Pi_i^{(0)}$ as a function of the π_i s and X^i s. Recalling finally that $\Pi_i = \Pi_i^{(1)} + \mathcal{O}(l_B^2)$ and using (22) we immediately get Π_i in terms of the π_i s and X^i s, $\Pi_i = \Pi_i(\boldsymbol{\pi}, \mathbf{X})$. The inversion of this relation, order by order in l_B , allows us to get π_i and x^i in terms of the adiabatic operators. The computation gives

$$\pi_i = \frac{1}{2} \{ \beta^{-1j}_i, \Pi_j \} + \frac{l_B}{2} c_{jh}^{klmn} \{ \beta^{-1j}_i \beta_m^o \Gamma_{on}^h, \{ \Pi_k, \Pi_l \} \} + \mathcal{O}(l_B^2) \quad (26)$$

$$x^i = X^i - l_B \varepsilon^{kl} \beta_k^i \Pi_l + \mathcal{O}(l_B^2) \quad (27)$$

where $c_{ij}^{klmn} = \frac{1}{2} \delta_i^n \delta_j^k \varepsilon^{ml} - 2c_{ij}^{klmn}$. As a useful check the commutation relations (14) may be re-obtained by means of (24).

The substitution of (26) and (27) in the Pauli Hamiltonian (13) yields immediately the first two terms of the adiabatic expansion (25),

$$\mathcal{H}^{(0)}/\hbar\omega_B = \frac{1}{2}\beta^{-1k}\beta^{-1l}\Pi_k\Pi_l + gb_i\sigma_i \quad (28)$$

$$\mathcal{H}^{(1)}/\hbar\omega_B = \beta^{-1p}\beta^{-1q}c_{pj}^{klmn}\beta_m^o\Gamma_{on}^j\{\Pi_k\Pi_q\Pi_l\} - g\varepsilon^{kl}\beta_k^h(\partial_h b_i)\sigma_i\Pi_l \dots \quad (29)$$

where the notation $\{\Pi_k\Pi_q\Pi_l\} = \Pi_k\Pi_q\Pi_l + \Pi_l\Pi_q\Pi_k$ has been introduced. In order to get some more physical insight into these expressions we now abandon our compact notation in favour of a more transparent one. By recalling the definition (15) of $\beta_i^j(\mathbf{x})$, definition (19) of $\Gamma_{ij}^k(\mathbf{x})$ and the explicit expression of the inhomogeneous dilatation $D_i^j(\mathbf{x}) = \text{diag}(b^{1/2}(\mathbf{x}), b^{1/2}(\mathbf{x}), 1)$, we rewrite everything in terms of the magnetic field and of other quantities capable of a direct physical interpretation. The full expansion of the zero-order Hamiltonian (28) gives

$$\mathcal{H}^{(0)}/\hbar\omega_B = \frac{1}{2}\Pi_3^2 + b[J + g(\mathbf{e}_3 \cdot \boldsymbol{\sigma})] \quad (30)$$

where J represents the harmonic oscillator Hamiltonian constructed by means of the canonical variables Π_1 and Π_2 , $J = (\Pi_1^2 + \Pi_2^2)/2$. The norm of the magnetic field $b(\mathbf{X})$ is evaluated in the adiabatic guiding centre operators \mathbf{X} . We observe that while the Π_1 – Π_2 degree of freedom decouples to this order from the slow and very slow variables the spin does not. The separation, up to higher-order terms, of the fast motion (rotation + spin) requires in fact a subsidiary zero-order transformation which we will perform in the next section. For the moment let us observe that, up to the spin term, the zero-order Hamiltonian (30) precisely embodies the expected behaviour of the system: the canonical pair of operators Π_1 – Π_2 takes into account the *fast* rotation of the particle around its guiding centre, while the non-canonical variables Π_3 – X^3 describe the slow motion along the magnetic field lines by means of an effective ‘kinetic energy + potential energy’ Hamiltonian. The norm of the magnetic field $b(\mathbf{X})$ plays the role of the effective potential. As long as $\mathcal{O}(l_B^2)$ terms are ignored the very slow dynamical variables X^1 – X^2 appear only as adiabatic parameters driving the slow motion, whereas a more accurate analysis indicates them as taking into account the very slow drift in the directions normal to the field [6].

The full expression of the first-order Hamiltonian (29) appears more complicated. The replacement of $\beta_i^j(\mathbf{x})$ and $\Gamma_{ij}^k(\mathbf{x})$ by means of (15) and (19) yields in fact the expression

$$\begin{aligned} \mathcal{H}^{(1)}/\hbar\omega_B = & -b^{-1/2}\varepsilon^{\mu\nu}(\mathbf{e}_\mu \cdot \nabla b)[\frac{2}{3}J_\nu + g(\mathbf{e}_3 \cdot \boldsymbol{\sigma})\Pi_\nu] - \frac{2}{3}b^{1/2}(\mathbf{e}_\mu \cdot \boldsymbol{\mathcal{A}})J_\mu \\ & + [\frac{1}{2}(\mathbf{e}_1 \cdot \mathbf{l}_2 - \mathbf{e}_2 \cdot \mathbf{l}_1) - (\mathbf{e}_3 \cdot \boldsymbol{\mathcal{A}})]J\Pi_3 + \frac{1}{4}(\mathbf{e}_1 \cdot \mathbf{l}_2 + \mathbf{e}_2 \cdot \mathbf{l}_1)(\Pi_1^2 - \Pi_2^2)\Pi_3 \\ & - \frac{1}{4}(\mathbf{e}_1 \cdot \mathbf{l}_1 - \mathbf{e}_2 \cdot \mathbf{l}_2)\{\Pi_1, \Pi_2\}\Pi_3 + b^{-1/2}[(\mathbf{e}_3 \cdot \mathbf{l}_2)\Pi_1 - (\mathbf{e}_3 \cdot \mathbf{l}_1)\Pi_2]\Pi_3^2 \\ & - gb^{1/2}\varepsilon^{\mu\nu}[(\mathbf{e}_\mu \cdot \mathbf{l}_1)(\mathbf{e}_1 \cdot \boldsymbol{\sigma}) + (\mathbf{e}_\mu \cdot \mathbf{l}_2)(\mathbf{e}_2 \cdot \boldsymbol{\sigma})]\Pi_\nu \end{aligned} \quad (31)$$

indicating the first-order coupling among the various operators. The notation $J_\mu = \frac{1}{2}\delta^{\alpha\beta}\Pi_\alpha\Pi_\mu\Pi_\beta$ has been introduced and all the functions are evaluated in \mathbf{X} . As expected from dimensional considerations $\mathcal{H}^{(1)}$ depends only on the first-order derivatives of the field. It is nevertheless worthwhile to stress that the gradient of the magnetic-field-norm, $\text{grad } b = \nabla b$, appears only in the first term of the right-hand side of this expression. All the remaining terms depend only on the quantities $\boldsymbol{\mathcal{A}}$, \mathbf{l}_1 and \mathbf{l}_2 completely characterizing the intrinsic/extrinsic geometry of the magnetic line bundle \mathcal{M} . To a large extent, therefore, the complication of this expression is produced by the variation of direction of the magnetic field, that is, by the nontrivial geometry of \mathcal{M} . It is not yet time to comment on the structure of $\mathcal{H}^{(1)}$. First of all, it is in fact necessary to operate a suitable unitary transformation separating the zero-order fast motion from the other degrees of freedom, that is diagonalizing

the the spin term $e_3 \cdot \sigma$. This will produce a modification of the first-order term of the adiabatic expansion. Secondly, it is possible to drastically simplify the form of $\mathcal{H}^{(1)}$ by operating a suitable first-order unitary transformation. The strategy is nothing but the quantum equivalent of the so-called *averaging transformation* of classical mechanics and sheds light on the physical content of (31).

6. Quantum averaging transformations

A well known strategy in dealing with the adiabatic separation of fast and slow variables in classical mechanics consists in performing a series of successive canonical transformations (the *averaging transformations*) separating, order by order in some adiabatic parameter, the rapid oscillation of the system from its slow averaged motion. As the analysis depends essentially on the canonical structure of the problem, it generalizes immediately to quantum mechanics, the canonical transformations being replaced by suitable unitary transformations. The full adiabatic expansion describing the motion of a spin degree of freedom adiabatically driven by external parameters has been obtained along these lines by Berry [16]. Littlejohn and Weigert [4] employed the method in discussing the first adiabatic corrections to the semiclassical motion of a neutral spinning particle in an inhomogeneous magnetic field. We shall consider therefore a set of unitary operators

$$U^{(n)} = \exp\{i l_B^n \mathcal{L}^{(n)}\} \tag{32}$$

$n = 0, 1, \dots$ such that fast and slow/very slow degrees of freedom separate up to $\mathcal{O}(l_B^{n+1})$ in the Hamiltonian obtained by the successive application of $U^{(0)}, U^{(1)}, \dots, U^{(n)}$. Whereas in classical mechanics it is natural to consider the averaging transformation as defining new canonical variables, in quantum mechanics it appears more convenient to keep the canonical operators fixed and transform the Hamiltonian.

6.1. Zero-order transformation

The zero-order separation of the fast and slow/very slow motion requires the diagonalization of the spin term $g b(\mathbf{X})(e_3(\mathbf{X}) \cdot \sigma)$ of Hamiltonian (30). Denoting by $\rho_i^j(\mathbf{x})$ the infinitesimal generator of the rotation $R_i^j(\mathbf{x})$ bringing the fixed frame $\{\hat{x}, \hat{y}, \hat{z}\}$ into the adapted frame $\{e_1(\mathbf{x}), e_2(\mathbf{x}), e_3(\mathbf{x})\}$, $R_i^j = (e^\rho)_i^j \equiv \delta_i^j + \rho_i^j + \frac{1}{2} \rho_i^k \rho_k^j + \dots$, the aim is achieved by choosing

$$\mathcal{L}^{(0)} = -\frac{1}{2} \varepsilon^{ijk} \rho_{ij}(\mathbf{X}) \sigma_k \tag{33}$$

the matrix $\rho_{ij} = \rho_i^j$ being evaluated in the guiding centre operators \mathbf{X} . Because of the commutation relations (24) the operator $U^{(0)}$ commutes with Π_1, Π_2 and therefore with J . It produces $\mathcal{O}(l_B)$ terms when commuting with Π_3 and $\mathcal{O}(l_B^2)$ terms when commuting with functions of \mathbf{X} . In evaluating the new Hamiltonian $\mathcal{H}' = U^{(0)} \mathcal{H} U^{(0)\dagger} = \mathcal{H}^{(0)'} + l_B \mathcal{H}^{(1)'} + \dots$ up to terms of order l_B^2 we have only to worry about the action of $U^{(0)}$ on σ and Π_3 . A very rapid computation yields the transformation rule

$$U^{(0)}(e_i \cdot \sigma) U^{(0)\dagger} = \sigma_i + \mathcal{O}(l_B^2) \tag{34}$$

while the action of $U^{(0)}$ on Π_3 , $U^{(0)} \Pi_3 U^{(0)\dagger} = \Pi_3 + U^{(0)} [\Pi_3, U^{(0)\dagger}]$, may be easily evaluated by computing the commutator in the original set of operators π_i s and x^i s and transforming back to adiabatic variables

$$U^{(0)} \Pi_3 U^{(0)\dagger} = \Pi_3 + l_B (e_3 \cdot l_2) \sigma_1 - l_B (e_3 \cdot l_1) \sigma_2 + l_B (e_3 \cdot \mathcal{A}) \sigma_3 + \mathcal{O}(l_B^2). \tag{35}$$

Subjecting $\mathcal{H}^{(0)}$ and $\mathcal{H}^{(1)}$ to the zero-order averaging transformation $U^{(0)}$ and by using (34) and (35) we obtain the new adiabatic expansion

$$\mathcal{H}^{(0)'} / \hbar\omega_B = \frac{1}{2}\Pi_3^2 + b(J + g\sigma_3) \quad (36)$$

$$\begin{aligned} \mathcal{H}^{(1)'} / \hbar\omega_B = & -b^{-1/2}\varepsilon^{\mu\nu}(e_\mu \cdot \nabla b)(\frac{2}{3}J_\nu + g\sigma_3\Pi_\nu) - \frac{2}{3}b^{1/2}(e_\mu \cdot \mathcal{A})J_\mu \\ & + [\frac{1}{2}(e_1 \cdot l_2 - e_2 \cdot l_1)J - (e_3 \cdot \mathcal{A})(J - \sigma_3)]\Pi_3 \\ & + \frac{1}{4}(e_1 \cdot l_2 + e_2 \cdot l_1)(\Pi_1^2 - \Pi_2^2)\Pi_3 - \frac{1}{4}(e_1 \cdot l_1 - e_2 \cdot l_2)\{\Pi_1, \Pi_2\}\Pi_3 \\ & + b^{-1/2}[(e_3 \cdot l_2)\Pi_1 - (e_3 \cdot l_1)\Pi_2]\Pi_3^2 + [(e_3 \cdot l_2)\sigma_1 - (e_3 \cdot l_1)\sigma_2]\Pi_3 \\ & - gb^{1/2}\varepsilon^{\mu\nu}[(e_\mu \cdot l_1)\sigma_1 + (e_\mu \cdot l_2)\sigma_2]\Pi_\nu \dots \end{aligned} \quad (37)$$

All the functions are evaluated in \mathbf{X} . The fast and slow/very slow motions are separated in this way in the zero-order term of the adiabatic expansion but not in the first-order term.

6.2. First-order transformation

The application of the first-order averaging transformation $U^{(1)}$ to \mathcal{H}' produces the new Hamiltonian $\mathcal{H}'' = U^{(1)}\mathcal{H}'U^{(1)\dagger} = \mathcal{H}^{(0)'} + l_B(\mathcal{H}^{(1)'} + i[\mathcal{L}^{(1)}, \mathcal{H}^{(0)'}]) + \dots$. It is then possible to simplify the first-order term of the adiabatic expansion by choosing $\mathcal{L}^{(1)}$ in such a way that its commutator with $\mathcal{H}^{(0)'}$ cancels as many terms of $\mathcal{H}^{(1)'}$ as possible. The analysis of the commutation relation involved and a little thought indicates that it is possible to annihilate all but the third term of (37) by choosing

$$\begin{aligned} \mathcal{L}^{(1)} = & -b^{-3/2}(e_\mu \cdot \nabla b)(\frac{2}{3}J_\mu + g\sigma_3\Pi_\mu) + \frac{2}{3}b^{-1/2}\varepsilon^{\mu\nu}(e_\mu \cdot \mathcal{A})J_\nu \\ & - \frac{1}{8}b^{-1}(e_1 \cdot l_2 + e_2 \cdot l_1)\{\Pi_1, \Pi_2\}\Pi_3 - \frac{1}{8}b^{-1}(e_1 \cdot l_1 - e_2 \cdot l_2)(\Pi_1^2 - \Pi_2^2)\Pi_3 \\ & - b^{-3/2}[(e_3 \cdot l_2)\Pi_2 + (e_3 \cdot l_1)\Pi_1]\Pi_3^2 + g^{-1}b^{-1}[(e_3 \cdot l_2)\sigma_2 + (e_3 \cdot l_1)\sigma_1]\Pi_3 \\ & + \frac{g}{g^2 - 4}b^{-3/2}[(e_\mu \cdot l_1)(2\sigma_1\delta^{\mu\nu} - g\sigma_2\varepsilon^{\mu\nu}) - (e_\mu \cdot l_2)(2\sigma_2\delta^{\mu\nu} + g\sigma_1\varepsilon^{\mu\nu})]\Pi_\nu. \end{aligned} \quad (38)$$

The commutators of the zero-order Hamiltonian (36) with the various terms of $\mathcal{L}^{(1)}$ yields the terms of (37) times the imaginary factor i , in such a way that they cancel in the new adiabatic expansion. To this are subjected all the terms but the third. It is in fact clear that no operator may be found in such a way that its commutator with (36) produces a term proportional to $J\Pi_3$ and $\sigma_3\Pi_3$. The third term of (37) may therefore not be removed from the adiabatic expansion representing the real first-order coupling among fast and slow/very slow motions and not a complication produced by a wrong choice of variables. Its relevance in the context of the classical guiding centre motion has been first recognized by Littlejohn [1]. It is therefore not a surprise to rediscover it in the discussion of the quantum guiding centre dynamics. The quantum averaging method therefore produces the adiabatic expansion

$$\mathcal{H}^{(0)''} = \mathcal{H}^{(0)'} \quad (39)$$

$$\mathcal{H}^{(1)''} / \hbar\omega_B = [\frac{1}{2}(e_1 \cdot l_2 - e_2 \cdot l_1)J - (e_3 \cdot \mathcal{A})(J - \sigma_3)]\Pi_3 \dots \quad (40)$$

We observe that whereas the zero-order terms (36) depend only on the magnetic-field-norm b (other than on the commutation relations (24)) the first-order term (40) is completely characterized by the Frobenius invariant (12), and by the magnetic field lines torsion (11).

7. Quantum guiding centre dynamics and magnetic-induced geometric-magnetism

The construction of a suitable set of non-canonical operators embodying the classically expected features of the motion of a charged particle in an inhomogeneous magnetic field and the quantum averaging method allow us to rewrite the Pauli Hamiltonian (13) in such a way that the fast degrees of freedom separate (up to terms of order l_B^2) from the guiding centre dynamics. The transformation to the adiabatic operators Π_i s, X^i s, (20) and (22), and the application of the zero- and first-order quantum averaging operators, (33) and (38), produces in fact the Hamiltonian

$$\mathcal{H}/\hbar\omega_B = \frac{1}{2}\Pi_3^2 + b(J + g\sigma_3) - l_B[\tau(J - \sigma_3) - \frac{1}{2}\mathcal{F}J]\Pi_3 + \mathcal{O}(l_B^2). \quad (41)$$

Disregarding terms of order higher than l_B the operators J —representing the magnetic moment of gyration of the particle—and σ_3 are constants of motion of the system. Frozen the particle in one of its J and σ_3 eigenstates Hamiltonian (41) describes therefore the corresponding guiding centre dynamics. As long as $\mathcal{O}(l_B^2)$ are ignored X^1 and X^2 appear as non-dynamical external adiabatic parameters and only the Π_3 – X^3 degree of freedom, representing in the classical limit the drift of the particle along the magnetic field lines, is dynamically relevant. To this order, therefore, the quantum guiding centre dynamics is described by a one-degree-of-freedom Hamiltonian given by the sum of the kinetic energy $\Pi_3^2/2$ and of an effective potential proportional to $b(\mathbf{X})$. As Π_3 is a *slow* variable, that is of the same magnitude of the first adiabatic correction, the order l_B term $[\tau(J - \sigma_3) - \mathcal{F}J/2]\Pi_3$ may be identified with a magnetic-like interaction and re-absorbed in the zero-order Hamiltonian as a gauge potential. The guiding centre Hamiltonian can be rewritten in this way in the familiar form

$$\mathcal{H}/\hbar\omega_B = \frac{1}{2}(\Pi_3 - l_B A(\mathbf{X}))^2 + V(\mathbf{X}) + \mathcal{O}(l_B^2) \quad (42)$$

with

$$A(\mathbf{X}) = (J - \sigma_3)\tau(\mathbf{X}) - \frac{J}{2}\mathcal{F}(\mathbf{X}) \quad (43)$$

$$V(\mathbf{X}) = (J + g\sigma_3)b(\mathbf{X}). \quad (44)$$

As it might be expected from the general discussion of section 3 the magnetic field line torsion $\tau = \mathbf{e}_3 \cdot \mathcal{A}$ appears as (a part of) a gauge potential in the effective slow dynamics, taking into account the anholonomy produced by the non-trivial parallel transport of the magnetic line bundle \mathcal{M} . Perhaps unexpected, at least from this point of view, is the contribution given by the Frobenius invariant. Let us in fact compare the guiding centre motion of a charged particle along a magnetic field line with the propagation of light in a coiled optical fibre [19] or with the motion of an electron constrained on a twisted line [20]. In both cases—sharing the same mathematical background—the adiabatic dynamics is coupled with an effective vector potential proportional to the torsion of the optical fiber or of the twisted line. The analogous contribution appears in the guiding centre motion, $(J - \sigma_3)\tau(\mathbf{X})$, but it is not the whole story. The particle not being homogeneously constrained in the neighbourhood of a line, it sensibly results in the variation of the geometry in the magnetic field lines surrounding the one on which the guiding centre is located. If all the field lines had the same geometry, the foliation of R^3 in terms of them would be trivial, the Frobenius invariant zero and the situation analogous the examples above. The geometry of this foliation is in general non-trivial as it yields a further contribution to the gauge potential $A(\mathbf{X})$ proportional to the Frobenius invariant, $J\mathcal{F}(\mathbf{X})/2$. It is obviously not possible, in the general case, to remove the gauge potential (43) by means of a suitable choice of gauge.

In order to make the identification of (42) with a one-degree-of-freedom Hamiltonian (two, if we want to consider X^1, X^2 as dynamical) complete, it is necessary to replace the Π_i s, X^i s by a set of canonical operators. The task is achieved by introducing a Darboux coordinate frame $\mathbf{x}^i = x^i(\mathbf{x})$, $i = 1, 2, 3$, bringing the closed skew-symmetric 2-form $b_{ij}(\mathbf{x})$ in the canonical form,

$$b_{kl}(\mathbf{x}) \frac{\partial x^k}{\partial x^i} \frac{\partial x^l}{\partial x^j} = \varepsilon_{ij} \quad (45)$$

$(x^1(\mathbf{x}), x^2(\mathbf{x}))$ may be identified with a pair of Euler potentials [18] for the magnetic field $\mathbf{b}(\mathbf{x})$, $\nabla_{\mathbf{x}^1} \wedge \nabla_{\mathbf{x}^2} = \mathbf{b}$, while $x^3(\mathbf{x})$ with the arc length of the magnetic field lines). Defining $X^i = x^i(\mathbf{X})$, $i = 1, 2, 3$ we get the canonical commutation relations $[\Pi_3, X^i] = -i l_B \delta_3^i$ and $[X^i, X^j] = -i l_B^2 \varepsilon^{ij}$ allowing the identification of the operators describing the slow and the very slow degrees of freedom. It is in principle possible to start from the beginning by introducing such curvilinear coordinates in R^3 and working out the problem by using a canonical set of operators [17]†. Nevertheless, whereas the existence of a Darboux coordinate frame is always guaranteed by the Darboux theorem, it is hardly ever possible to find it explicitly and to proceed to the construction of the X^i s. For this reason—though the Π_i s, X^i s appear as the most natural variables for the problem—the explicit construction of a set of non-canonical operators appears a better strategy.

8. Conclusions

The main difficulty in addressing the separation of fast and slow degrees of freedom in the study of an adiabatic system consist generally in finding out a suitable set of variables adapting with sufficient accuracy to the different time scale behaviours of the system. Starting from the homogeneous case and the canonical commutation relations (5) we showed how the analysis of the canonical structure of a charged spinning particle moving in an external inhomogeneous magnetic field leads naturally to the construction—as power series in the magnetic length l_B —of a suitable set of non-canonical operators. This allows us to systematically take into account the coupling between spatial and spin degrees of freedom. The new variables fulfil the very compact commutation relations (24) clearly displaying the dependence of the canonical structure on the norm and direction of the external magnetic field. In terms of the new operators the Pauli Hamiltonian can be rewritten as a power series in the adiabatic parameter l_B which may be brought into a particular simple form by operating suitable unitary transformations. In this way the *fast* degree of freedom of the system is separated from the remaining degrees of freedom up to terms of order l_B^2 . The resulting effective guiding centre dynamics displays geometric-magnetism: (i) the coupling with the geometry induced gauge potential (43), depending on the magnetic field lines torsion (11) and on the Frobenius invariant (12), (ii) the coupling with the scalar potential (44), proportional to the magnetic field norm. This completely extends to the quantum domain the previous classical treatments of the problem showing that the anholonomy first studied by Littlejohn in the classical guiding centre theory plays an equivalent role in the discussion of the quantum problem. It is a feature of the canonical structure of the system, after all. The geometrical mechanism responsible for the appearance of induced gauge structures has also been analysed in some detail and formalized in the geometry of the magnetic line bundle \mathcal{M} .

† The introduction of Darboux coordinate produces automatically the framing of $T_x R^3$ by means of an adapted frame $\{e_1(\mathbf{x}), e_2(\mathbf{x}), e_3(\mathbf{x})\}$. Our method, on the other hand, consists in adapting the frame of $T_x R^3$ without introducing the curvilinear coordinates.

In conclusion we observe that our discussion gives the solution of only half of the problem. The guiding centre dynamics is still characterized by the presence of a *fast* and a *slow* time scale (*slow* \rightarrow *fast*, *very slow* \rightarrow *slow*) and is therefore amenable to a treatment by means of adiabatic techniques. Nevertheless, the remaining problem is not of such a deep geometrical nature as the original one.

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